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On manifestly Sp(2) invariant formulation of quadratic higher spin lagrangians

K.B. Alkalaev

I.E. Tamm Department of Theoretical Physics, P.N. Lebedev Physical Institute, Leninsky ave. 53, 119991 Moscow, Russia,[*] and
Scuola Normale Superiore, Piazza dei Cavalieri 7, I-56126 Pisa, Italy,[†] and
INFN, Piazza dei Cavalieri 7, I-56126 Pisa, Italy
E-mail: alkalaev@lpi.ru

Abstract: The Lagrangian frame-like formulation of free higher spin symmetric bosonic AdS_d fields is given within a manifestly $sp(2)$ invariant framework. It is designed to deal with infinite multiplets of fields appearing as gauge connections of the higher spin algebras.

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[*] Permanent address.
[†] Postdoctoral visitor of INTAS and Scuola Normale Superiore.

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1. Introduction

At the free field level, the dynamics of higher spin (HS) symmetric massless fields is pretty well understood both on the flat and $(A)dS_d$ backgrounds. Various formulations of HS symmetric field dynamics are available but basically there are two main methods to describe HS fields, the metric-like and the frame-like approaches. Within the more traditional metric-like formulation HS fields are described by Lorentz-covariant tensor fields [1 − 7].[1] The frame-like formulation describes HS fields as p-forms with tangent Lorentz indices of definite symmetry types [9 − 12]. Both formulations are dynamically equivalent and the metric-like fields result from the frame-like ones by virtue of partial gauge fixing. There exists also the so called parent theory which encodes these two forms of HS field dynamics and particular realizations can be reached by one or another reduction [13].

[1] The light-cone actions for arbitrary AdS_5 mixed-symmetry massless fields were constructed in [8].

On the $(A)dS_d$ backgrounds the HS fields may exhibit an interesting property of partial masslessness [14 − 18]. HS fields of this type possess a reduced gauge symmetry compared to that of massless fields and describe either non-unitary dynamics (for the AdS_d background) or dynamics with the energy not bounded from below (for the dS_d background). In the flat limit partially massless fields do not exist and reduce to usual massless fields.

From the group-theoretical point of view, a given formulation of a single free field should give rise to the corresponding infinite-dimensional representation of the algebra of global space-time symmetries. More precisely, the space of one-particle states should form a unitary representation of the Poincare algebra or the $(A)dS_d$ algebra with the energy bounded from below. The conditions of unitarity or bounded energy can be relaxed like in the case of partially massless fields.

From the perspective of the higher spin interaction problem, a free field theory is required to satisfy some additional conditions. The reason is that the higher spin interactions are governed by a higher spin algebra which describes both global and gauge higher spin symmetries. It defines a field content of the theory and consistent deformations of linearized gauge symmetries. This implies that for a free field theory to be a limit of some non-linear theory the fields must be organized into an infinite higher spin multiplet. In other words, fields of a given higher spin multiplet form a representation of the algebra of global higher spin symmetries. This is the so called admissibility condition [19].

In this paper we aim to develop a Lagrangian framework for free higher spin dynamics that naturally operates with infinite sets of (partially) massless bosonic symmetric fields considered as gauge connections of the higher spin algebras. Our goal is motivated by a desire to develop a Lagrangian description of the higher spin couplings in the cubic order and beyond.[2]

Our considerations rest on recent developments in the non-linear HS theory utilizing the unfolded form of HS dynamics [20 − 22]. The main technical ingredient proposed in [20] is the use of $sp(2)$ symmetry as internal symmetry in the auxiliary space. More precisely, the unfolded HS dynamics is formulated in terms of functions $F(x|\,Y)$, which depend on the spacetime coordinates x^n and $sp(2)$ doublets of the auxiliary $o(d-1,2)$ vector variables Y^A_α and are subject to the $sp(2)$ invariance condition. It follows that the fields of the theory are identified with the expansion coefficients with respect to the auxiliary variables. It is remarkable that the use of the auxiliary $sp(2)$ symmetry brings together the previously known unfolded field equations and the newly defined HS algebra, and provides for them a unified framework.[3] Having in mind a Lagrangian form of non-linear dynamics, these results along with the Lagrangian frame-like formulation [11, 12] provide a good starting point.

[2]The analogous problems for non-symmetric (partially) massless fields require a separate study. The reason is that higher spin algebras with spectra of AdS_d fields with any spins are unknown yet [21]. Let us also note that the frame-like formulation for arbitrary non-symmetric massless fields in AdS_d is available now [27].

[3]Within the unfolded formulation one may consider two different versions of the $sp(2)$ invariance condition, a weak condition originally introduced in [20], and a strong one subsequently proposed in [22]. The strong form of $sp(2)$ invariance requires further investigations and should be treated with a great care when one considers interactions. In the present paper, we study free gauge fields only and use the $sp(2)$ invariance condition in the form proposed in [20].

Let us recall now the general properties of the HS algebra that describes massless symmetric bosonic AdS_d fields of any spins from zero to infinity [23, 20]. By definition, it is a quotient algebra and massless HS fields are identified with the representatives of the equivalence classes. More precisely, HS algebra is a quotient S/I, where S is an infinite-dimensional Lie algebra endowed with the Weyl $*$-product commutator, which describes two-row $o(d-1,2)$ traceful tensors, and I is a two-sided ideal generated by traces. It follows that the $*$-product of any two elements satisfying the tracelessness condition does not necessarily satisfies the same condition (otherwise these elements would form a subalgebra rather than a quotient algebra). It causes the problem of explicit realization of the $*$-product on the factor space. In particular, the structure constants of the HS algebra are not known yet in general. However, there is a nice projection technique based on the quasi-projector Δ that allows one to perform a factorization procedure automatically without an explicit calculation of particular representatives [20, 21, 12, 25, 22].

The algebra S describes infinitely degenerate sets of massless and partially-massless fields, while the quotient S/I describes massless fields only and each field enters in a single copy. The factorization procedure removes partially massless fields and reduces an infinite degeneracy of massless fields. In principle, a non-linear theory governed by the algebra S may be of interest. It would provide an example of the gauge system with an infinitely extended gauge symmetry that contains conventional massless HS gauge theories. In particular, this point of view leads to the issue of consistent interactions of partially massless HS fields with the gravity and between themselves.[4] Let us emphasize, that a presence of partially-massless fields extends gauge symmetry of the theory at the expense of a lack of untarity. However, it still makes sense to study such extended theories since unitary massless theory could be embedded into it by virtue of one or another scenario.

In this paper we propose to implement the projection technique described above on the level of action functionals. The procedure has two stages. Firstly, one introduces the action functional $\mathcal{S}[\Omega]$ defined on the fields Ω which are elements of the algebra S. Secondly, one builds the projector Δ into the action $\mathcal{S}$ in an appropriate fashion. The action $\mathcal{S}_\Delta[\Omega]$ equipped with the projector is formally defined on the elements of the algebra S but the presence of the projector reduces it to the quotient S/I. In fact, this approach is inspired by the analysis of [12, 25], where the similar projection procedure was used in the study of $5d$ HS cubic couplings.

We study action functionals defined on the fields taking values in the algebra S, while the actions defined on the quotient S/I will be given elsewhere [26]. From the technical point of view, when analyzing action functionals we use a particular framework of [27] developed to describe a frame-like form of mixed-symmetry field dynamics. It is remarkable that the approach of [27] fits naturally the definition of HS algebra [20, 21] thus making the analysis of the problem feasible.

The paper is organized as follows. In section 2 we shortly review frame-like formulation of symmetric bosonic fields in d dimensions. Section 3 contains a discussion of $sp(2)$

[4]It was shown that consistent cubic vertex of a partially massless spin-5/2 coupled to the AdS_d gravity does exist [24].

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doublets of $o(d-1,2)$ vector variables and the trace decomposition of rectangular traceful $o(d-1,2)$ tensors. In section 4 we introduce the bilinear form defined on arbitrary $sp(2)$ singlet fields, which serves as a basis for the action functionals discussed in section 5. In particular, in section 5.3 we build HS action that describes an infinite set of massless gauge fields with spins $2 \leq s \leq \infty$ and reproduces known expression for the component form of HS action elaborated in ref. [12]. Finally, we present our conclusions in section 6. In appendix we collect and discuss some useful formulae used in the calculations.

2. A summary on HS symmetric fields

Let us briefly recall some basic facts on the frame-like formulation of HS symmetric fields. More detailed expositions can be found, for instance, in [28] and [18].

Within the metric-like approach an integer spin-s massless field is described by a totally symmetric rank-s $o(d-1,1)$ tensor $\varphi^{a_1\cdots a_s}$ subject to the Fronsdal double tracelessness condition [1]. The higher spin gauge transformations read $\delta\varphi^{a_1\cdots a_s} = \mathcal{D}^{(a_1}\varepsilon^{a_2\cdots a_s)}$, where the parameter $\varepsilon^{a_1\cdots a_{s-1}}$ is a rank $(s-1)$ symmetric traceless tensor and $\mathcal{D}^a$ is the background Lorentz derivative.[5]

The frame-like formalism operates with 1-form field [12]

$$\Omega^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}} = \mathrm{d}x^{\underline{n}}\,\Omega_{\underline{n}}{}^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}} \tag{2.1}$$

that carries the traceless tensor representation of $o(d-1,2)$ described by the length $s-1$ two-row rectangular Young tableau, i.e., symmetrization of any s tangent indices of (2.1) gives zero.

With the 1-form field (2.1) one associates the linearized curvature

$$R^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}} = D_0\Omega^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}}\,, \tag{2.2}$$

where $D_0T^A = \mathrm{d}T^A + \Omega_0^A{}_B T^B$ is the $o(d-1,2)$ covariant derivative evaluated with respect to the AdS_d background 1-form connection $\Omega_0^{AB} = -\Omega_0^{BA}$ that satisfies the zero curvature equation $D_0D_0 = \mathrm{d}\Omega_0^{AB} + \Omega_0^A{}_C \wedge \Omega_0^C{}_B = 0$. The last property implies that the curvature (2.2) is invariant under HS gauge transformations

$$\delta\Omega^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}} = D_0\varepsilon^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}} \tag{2.3}$$

with a traceless 0-form gauge parameter.

Being decomposed into $o(d-1,1)$ components, the field (2.1) yields a collection of 1-forms [11]

$$\omega^{a_1\dots a_{s-1},\,b_1\dots b_t}(x) = \mathrm{d}x^{\underline{n}}\,\omega_{\underline{n}}{}^{a_1\dots a_{s-1},\,b_1\dots b_t}(x)\,, \qquad 0 \leq t \leq s-1\,. \tag{2.4}$$

[5] We work within the mostly minus signature and use notation $\underline{m},\underline{n} = 0 \div d-1$ for world indices, $a,b = 0 \div d-1$ for tangent Lorentz $o(d-1,1)$ vector indices and $A,B = 0 \div d$ for tangent AdS_d $o(d-1,2)$ vector indices. We also use the condensed notation of [10] and denote a set of symmetric indices $(a_1\cdots a_s)$ as $a(s)$. All symmetrizations are performed with a unit weight, e.g. $X^{(a}Y^{b)} = X^aY^b + X^bY^a$.

The 1-form with $t=0$ is the physical field $\omega_{\underline{n}}{}^{a(s-1)}$. The 1-form with $t=1$ is the auxiliary field $\omega_{\underline{n}}{}^{a(s-1),\,b}$. The remaining 1-forms (2.4) with $t\geq 2$ are extra fields [10, 11]. The metric-like field is identified with a component of the physical field obtained by the symmetrization $\omega^{(a_1;\,a_2\dots a_s)}(x)=\varphi^{a_1\dots a_s}(x)$.

Lorentz fields play different dynamical roles depending on the values of parameter t. For example, in the spin two case the decomposition has the form $\Omega^{AB}\rightarrow\omega^a\oplus\omega^{ab}$, where ω^a is the vielbein (the physical field) and $\omega^{ab}=-\omega^{ba}$ is the Lorentz spin connection (the auxiliary field). On the level of the equations of motion the auxiliary field is expressed through the first derivatives of the physical field. Extra fields are absent in this case. They appear starting from a spin three field and are required to enter the action via total derivatives only. Their role is to maintain a manifest gauge invariance of the action built as a bilinear combination of linearized curvatures (2.2) (for more details, see ref. [28] and sections 4, 5 of the present paper).

The decomposition procedure of $o(d-1,2)$ covariant fields into their $o(d-1,1)$ irreducible components can be done in a manifestly $o(d-1,2)$ covariant fashion [29]. To this end one identifies the Lorentz algebra as a stability subalgebra of the compensator vector V^A normalized as $V^AV_A=1$. Then the resulting set of Lorentz components can be described as $o(d-1,2)$ tensors which are orthogonal to the compensator vector.

In particular, the component of $\Omega^{A(s-1),\,B(s-1)}$, that is most parallel to the compensator V^A, is the physical field $\omega^{A(s-1)}=\Omega^{A(s-1),B(s-1)}V_B\cdots V_B$. (In order to obtain manifestly Lorentz covariant expressions it is convenient to substitute the compensator in the form $V^A=\delta^A{}_d$.) The less V^A-longitudinal components are identified with the other fields in the set (2.4).

It is useful to introduce a field

$$\hat{\Omega}^{A_1\dots A_{s-1},\,B_1\dots B_p}\equiv\Omega^{A_1\dots A_{s-1},\,B_1\dots B_pC_{p+1}\dots C_{s-1}}\;V_{C_{p+1}}\cdots V_{C_{s-1}}\;,\quad p\leq s-1\;,\tag{2.5}$$

defined as a contraction of the original field (2.1) with a number of compensators.[6] It decomposes into a set of Lorentz fields (2.4) with $0\leq t\leq p$. In the case of $p=1$ the field $\hat{\Omega}^{A(s-1),\,B}$ contains just two Lorentz components which are identified with the physical ($t=0$) and the auxiliary ($t=1$) fields.

The $o(d-1,2)$ covariant versions of the background frame and background Lorentz spin connection are defined as follows [29]

$$\lambda E_0^A=D_0\,V^A\equiv \mathrm{d}V^A+\Omega_0^{AB}V_B\;,\quad \omega_0^{AB}=\Omega_0^{AB}-\lambda\,(E_0^A\,V^B-E_0^B\,V^A)\;.\tag{2.6}$$

Here the parameter λ is the inverse radius of the AdS_d space.

The frame-like formulation of HS dynamics provides a description for a wider class of relativistic fields that may propagate on the $(A)dS_d$ backgrounds known as partially massless fields [15]. Indeed, according to ref. [18] a partially-massless field of spin s and depth t can be described by 1-form

$$\Omega^{A_1\dots A_{s-1},\,B_1\dots B_t}=\mathrm{d}x^{\underline{n}}\,\Omega_{\underline{n}}{}^{A_1\dots A_{s-1},\,B_1\dots B_t}\;,\tag{2.7}$$

[6] Let us note that by virtue of Young symmetry properties a contraction with s compensators gives zero. Also, any contraction of a rectangular Young tableau with the symmetric tensor $V_{A_1\dots A_k}=V_{A_1}\cdots V_{A_k}$ can be reduced to a contraction of $V_{A_1\dots A_k}$ with indices of the bottom row.

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that carries the traceless tensor representation of $o(d-1,2)$ described by two-row non-symmetric Young tableau with lengths of rows $s-1$ and t. The corresponding field-theoretical systems describe either non-unitary dynamics (for the AdS_d background) or dynamics with the energy not bounded from below (for the dS_d background).

The linearized curvatures and the gauge transformations are defined in the same manner as for massless fields

$$R^{A_1\ldots A_{s-1},\, B_1\ldots B_t} = D_0\Omega^{A_1\ldots A_{s-1},\, B_1\ldots B_t}\,, \tag{2.8}$$

$$\delta\Omega^{A_1\ldots A_{s-1},\, B_1\ldots B_t} = D_0\varepsilon^{A_1\ldots A_{s-1},\, B_1\ldots B_t}\,, \tag{2.9}$$

and reproduce the expressions (2.2) and (2.3) at $t=s-1$.

Gauge transformations for partially-massless fields written in a metric-like form involve higher derivatives up to $(s-t)$-th order [15]. It is worth to note that gauge transformations for frame-like fields, both massless and partially-massless, involve just one derivative, as is seen from (2.9).

In what follows we assume that the parameter t is free and runs from 0 to $s-1$ so both massless and partially-massless fields are considered on equal footing.

3. Non-symmetric tensors and Howe duality

In this section we discuss auxiliary $o(d-1,2)$ vector variables and their Howe dual $sp(2)$ algebra. Although much of the discussion is familiar from refs. [20, 21], we feel it is important to set notation carefully, as we will introduce some new ingredients.

3.1 Auxiliary $sp(2)$ covariant variables

Let Y^A_α be auxiliary commuting variables[7] with $A=0\div d$ and $\alpha=1,2$. Indices A are raised and lowered with the invariant symmetric form η_{AB} of $o(d-1,2)$ as $X_A=\eta_{AB}X^B$, indices α will be later on specified to $sp(2)$.

The expansion coefficients of polynomial of given orders m_1, m_2 in variables Y^A_1 and Y^B_2

$$F(Y) = F_{A_1\ldots A_{m_1};\, B_1\ldots B_{m_2}}\; Y_1^{A_1}\cdots Y_1^{A_{m_1}}\; Y_2^{B_1}\cdots Y_2^{B_{m_2}} \tag{3.1}$$

are $o(d-1,2)$ tensors that are symmetric in the indices A_i and B_j. To specify a particular Young symmetry type of indices one introduces operators

$$L_\alpha{}^\beta = Y^A_\alpha\,\frac{\partial}{\partial Y^A_\beta} \tag{3.2}$$

that form the $gl(2)$ algebra $[L_\alpha{}^\beta, L_\gamma{}^\rho] = \delta_\alpha{}^\rho L_\gamma{}^\beta - \delta_\gamma{}^\beta L_\alpha{}^\rho$, which is Howe dual symmetry [30, 20, 21]. Young symmetry conditions read then

$$L_\alpha{}^\beta\, F(Y)\Big|_{\alpha<\beta} = 0\,, \qquad L_\alpha{}^\beta\, F(Y)\Big|_{\alpha=\beta} = m_\alpha\, F(Y)\,, \tag{3.3}$$

[7] In the sequel we introduce additional set of auxiliary dotted variables $Y^A_{\dot\alpha}$, $\dot\alpha=1,2$. All constructions of this section for undotted variables Y^A_α are valid for dotted ones as well.

and mean that $F(Y)$ belongs to a highest weight representation of $gl(2)$ [30, 20, 21]. It results in a standard symmetrization condition for the expansion coefficients

$$F^{(A_1\dots A_{m_1};A_{m_1+1})B_2\dots B_{m_2}}=0\ . \tag{3.4}$$

An important observation is that Young symmetry conditions specifying a block tableau ($m_1=m_2$) can be reformulated as an invariance of $F(Y)$ under $sl(2)\subset gl(2)$ transformations [30, 20, 21]. By definition, $sl(2)$ generators are traceless parts of $gl(2)$ ones

$$\tilde{L}_\alpha{}^\beta=L_\alpha{}^\beta-\frac{1}{2}\delta_\alpha{}^\beta N\ ,\qquad [\tilde{L}_\alpha{}^\beta,\tilde{L}_\gamma{}^\rho]=\delta_\alpha{}^\rho\tilde{L}_\gamma{}^\beta-\delta_\gamma{}^\beta\tilde{L}_\alpha{}^\rho\ , \tag{3.5}$$

(here $N=L_\gamma{}^\gamma$ is a central element of $gl(2)$), so the $sl(2)$ invariance is expressed by the condition

$$\tilde{L}_\alpha{}^\beta F(Y)=0\ . \tag{3.6}$$

In other words, polynomial $F(Y)$ with coefficients being two-row rectangular tableaux is a $sl(2)$ singlet. Note that N plays the role of the Euler operator that counts a total number of the variables Y^A_α.

Now we recall the well-known fact that $2d$ Levi-Civita symbol $\epsilon_{\alpha\beta}$ is an invariant tensor of $sl(2)$ algebra. In particular, it makes the isomorphism $sl(2)\sim sp(2)$ clear. From now on we assume that indices α are raised and lowered with $sp(2)$ invariant antisymmetric form $\epsilon_{\alpha\beta}$ as $X^\alpha=\epsilon^{\alpha\beta}X_\beta$, $X_\alpha=X^\beta\epsilon_{\beta\alpha}$.

The generators of $sp(2)$ expressed via $sl(2)$ ones

$$T_{\alpha\beta}=\epsilon_{\alpha\gamma}\tilde{L}_\beta{}^\gamma+\epsilon_{\beta\gamma}\tilde{L}_\alpha{}^\gamma\equiv\epsilon_{\alpha\gamma}L_\beta{}^\gamma+\epsilon_{\beta\gamma}L_\alpha{}^\gamma \tag{3.7}$$

form the $sp(2)$ algebra $[T_{\alpha\beta},T_{\gamma\rho}]=-\epsilon_{\alpha\gamma}T_{\beta\rho}-\epsilon_{\alpha\rho}T_{\beta\gamma}-\epsilon_{\beta\gamma}T_{\alpha\rho}-\epsilon_{\beta\rho}T_{\alpha\gamma}$. Variables Y^A_α rotate as $sp(2)$ vectors

$$[T_{\alpha\beta},Y^A_\gamma]=\epsilon_{\alpha\gamma}Y^A_\beta+\epsilon_{\beta\gamma}Y^A_\alpha\ . \tag{3.8}$$

An equivalent form of (3.6) reads now

$$T_{\alpha\beta}F(Y)\equiv\Big(\epsilon_{\alpha\gamma}L_\beta{}^\gamma+\epsilon_{\beta\gamma}L_\alpha{}^\gamma\Big)F(Y)=0\ . \tag{3.9}$$

The analog of formula (3.5)

$$T_{\alpha\beta}=2L_\alpha{}^\gamma\epsilon_{\gamma\beta}-\epsilon_{\alpha\beta}N \tag{3.10}$$

is useful in calculations and is a simple consequence of rank-2 tensor decomposition into (anti)symmetric parts.

3.2 Trace decomposition

Consider a polynomial $F(Y)$ subject to $sp(2)$ invariance condition (3.9). The expansion coefficients of $F(Y)$ are two-row rectangular Young tableaux $F^{A_1\dots A_m;\,B_1\dots B_m}\equiv F^{A(m),\,B(m)}$. In general, tensors $F^{A(m),\,B(m)}$ are traceful.

To examine a decomposition of a traceful tensor $F^{A(m),B(m)}$ into traceless components one needs to study the symmetry properties of n-valued product of $o(d-1,2)$ invariant tensors η_{AB}. Coupling n traces to $F^{A(m),\,B(m)}$ means that one takes a symmetrized tensor product of n traces and then projects out the components with more than two rows. Graphically, this operation is represented as follows

$$\mathcal{P}\Big(\underbrace{\square\square\;\otimes\cdots\otimes\;\square\square}_{n\;\;times}\Big)_{\rm sym} = \bigoplus_{2l+k=n} \begin{array}{l}\overbrace{\square\square\square\square\square\square\square\square}^{2l+2k}\\ \underbrace{\square\square\square\square}_{2l}\end{array}\qquad, \tag{3.11}$$

where $\mathcal{P}$ is a projector on the two-row tensors arising in the tensor product. Obviously, one can take any number of non-trivial traces from 0 to $2\,[m/2]$. It follows that for an arbitrary two-row rectangular traceful tensor there is a two-parametric family of components

$$F^{A(m),\,B(m)} = \bigoplus_{l=0}^{[m/2]}\;\bigoplus_{k=0}^{[m/2]-l}\; F^{A(m-2l),\,B(m-2l-2k)}\;. \tag{3.12}$$

Here the parameter $2l+k=n$ counts a number of removed traces and two-row tensors in the right-hand-side are traceless

$$\eta_{A(2)}F^{A(p),\,B(t)} = 0\;, \quad \eta_{AB}F^{A(p),\,B(t)} = 0\;, \quad \eta_{B(2)}F^{A(p),\,B(t)} = 0\;. \tag{3.13}$$

Note that by virtue of Young symmetry (3.4) only the first condition is independent while the others are its linear combinations. For a block tableau, i.e. when $p=t$, the first and the last conditions are equivalent because of a block symmetry property $F^{A(m),\,B(m)} = (-)^m F^{B(m),\,A(m)}$.

We now turn to a reformulation of the above results within a manifestly $sp(2)$ covariant framework. It heavily rests on a possibility to describe non-symmetric two-row tensors as $sp(2)$ singlets. The only restriction is that a difference between the lengths of first and second rows must be even.

Let us introduce the operators

$$t_{\alpha\beta} = \eta_{AB}\,Y^A_\alpha Y^B_\beta \quad\text{ and }\quad \bar{s}^{\alpha\beta} = \eta^{AB}\,\frac{\partial^2}{\partial Y^A_\alpha \partial Y^B_\beta}\;. \tag{3.14}$$

Their commutation relation is given by

$$\begin{aligned}[\bar{s}^{\alpha\beta}, t_{\gamma\rho}] = \Big(\frac{N}{2}+d+1\Big)&(\delta_\gamma{}^\beta\delta_\rho{}^\alpha + \delta_\rho{}^\beta\delta_\gamma{}^\alpha) + \\ &+\frac{1}{2}(\delta_\gamma{}^\beta T_\rho{}^\alpha + \delta_\rho{}^\beta T_\gamma{}^\alpha + \delta_\gamma{}^\alpha T_\rho{}^\beta + \delta_\rho{}^\alpha T_\gamma{}^\beta)\;.\end{aligned} \tag{3.15}$$

Note also that the operators $t_{\alpha\beta}$, $\bar{s}^{\gamma\rho}$ and $l_\mu{}^\nu = L_\mu{}^\nu + \dfrac{d+1}{2}\,\delta_\mu{}^\nu$ form the $sp(4)$ algebra, which is the Howe dual algebra for traceless two-row tensors [30, 20, 21].[8]

[8]The space of polynomials $F(Y)$ carries representations of Howe dual algebras, $o(d-1,2)$ and $sp(4)$ (and $sp(2)\subset sp(4)$). Since these algebras commute the space $F(Y)$ decomposes into a direct sum of irreducible highest weight $sp(4)$ representations. A non-trivial statement proved in [13] is that for $d\geq 3$ each irreducible $sp(4)$ component is an infinite-dimensional generalized Verma module induced from a finite-dimensional $sp(2)$ representation and does not not contain singular vectors except the trivial one generated by the $sp(2)$ generator. I am grateful to M. Grigoriev for the illuminating discussions of this issue.

Since traceless $o(d-1,2)$ tensors satisfy the constraint

$$\bar{s}^{\alpha\beta}F(Y)=0\,, \tag{3.16}$$

it follows that $t_{\alpha\beta}$ and $\bar{s}^{\gamma\rho}$ act like a trace creation and a trace annihilation operators. They are symmetric tensors with respect to the $sp(2)$ transformations

$$[T_{\alpha\beta},t_{\gamma\rho}]=\epsilon_{\alpha\gamma}t_{\beta\rho}+\epsilon_{\beta\gamma}t_{\alpha\rho}+\epsilon_{\alpha\rho}t_{\beta\gamma}+\epsilon_{\beta\rho}t_{\alpha\gamma}\,, \tag{3.17}$$
$$[T_{\alpha\beta},\bar{s}^{\gamma\rho}]=\delta_{\beta}{}^{\rho}\bar{s}_{\alpha}{}^{\gamma}+\delta_{\alpha}{}^{\rho}\bar{s}_{\beta}{}^{\gamma}+\delta_{\beta}{}^{\gamma}\bar{s}_{\alpha}{}^{\rho}+\delta_{\alpha}{}^{\gamma}\bar{s}_{\beta}{}^{\rho}\,. \tag{3.18}$$

Consider now $o(d-1,2)$ tensors that are not traceless, i.e.,

$$\bar{s}^{\alpha\beta}F(Y)\neq 0. \tag{3.19}$$

Taking into account the commutation relation (3.15) one finds that the general solution of the condition (3.19) reads

$$F(Y)=F_0(Y)+t_{\alpha\beta}\,F_1^{\alpha\beta}(Y)\,, \tag{3.20}$$

where $F_0(Y)$ satisfies (3.16), while $F_1^{\alpha\beta}(Y)$ is a symmetric $sp(2)$ tensor that rotates as

$$[T^{\alpha\beta},F_1^{\gamma\rho}]=\epsilon^{\alpha\gamma}F_1^{\beta\rho}+\epsilon^{\beta\gamma}F_1^{\alpha\rho}+\epsilon^{\alpha\rho}F_1^{\beta\gamma}+\epsilon^{\beta\rho}F_1^{\alpha\gamma}\,, \tag{3.21}$$

thus a combination $t_{\alpha\beta}F_1^{\alpha\beta}$ remains invariant under the $sp(2)$ transformations [20].

Let us suppose that $\bar{s}^{\alpha\beta}F_1^{\gamma\rho}(Y)=0$. Applying the trace annihilation operator $\bar{s}^{\alpha\beta}$ to both sides of the relation (3.20) we obtain that

$$\bar{s}^{\alpha\beta}F(Y)=(N+2d+10)\,F_1^{\alpha\beta}(Y)\,,\qquad \bar{s}^{\alpha\beta}\bar{s}^{\gamma\rho}F(Y)=0\,. \tag{3.22}$$

These expressions imply that $F(Y)$ describes a double traceless $o(d-1,2)$ tensor. A peculiarity caused by a manifest $sp(2)$ covariance of the whole analysis consists in the illusory mismatch between a number of traceless components and a number of trace annihilation operators (3.14). Indeed, according to the trace decomposition (3.12), a double traceless two-row rectangular tensor $F^{A(m),\,B(m)}$ decomposes into traceless components as follows

$$F^{A(m),\,B(m)}=F_0^{A(m),\,B(m)}\oplus F_1^{A(m),\,B(m-2)}\,. \tag{3.23}$$

By comparing to (3.20) one identifies the first term in (3.23) with $F_0(Y)$ and the second one with $F_1^{\alpha\beta}(Y)$. Then one observes that the non-symmetric tensor corresponding to the first trace of F is described by three functions F_1^{11},F_1^{12} and F_1^{22} and not by a single one as one might expect. However, commutation relation (3.21) ensures that not all functions $F_1^{\alpha\beta}$ are independent and can be expressed in terms of just one, for instance, F_1^{22}. More precisely, particular relations $[T^{11},F^{22}]=4F^{12}$ and $[T^{11},F_1^{12}]=2F_1^{11}$ read off from (3.21) imply that one rotates F^{22} to obtain F^{11} and F^{12}. The expansion coefficients of function F^{22} are $F_1^{A(m),\,B(m-2)}$, while F^{12},F^{11} describe $F_1^{A(m-1)B,\,B(m-2)}$ and $F_1^{A(m-2)B(2),\,B(m-2)}$, respectively. The action of generator $T^{11}=-2L_2{}^1$ is in fact a proper symmetrization that lengthens a second row.

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The $sp(2)$ covariant description of more than double traceless tensors considered above is quite analogous. A general decomposition of order $2m$ polynomial $F(Y)$ to traceless parts reads

$$F(Y) = \sum_{n=0}^{2[m/2]} t_{\alpha_1\beta_1} \cdots t_{\alpha_n\beta_n} \, F_n^{\alpha_1\beta_1;\, \cdots ;\, \alpha_n\beta_n}(Y)\,, \tag{3.24}$$

$$\bar{s}^{\gamma\rho} \, F_n^{\alpha_1\beta_1;\, \cdots ;\, \alpha_n\beta_n}(Y) = 0\,.$$

The $sp(2)$ representation carried by $F_n^{\alpha_1\beta_1;\, \cdots ;\, \alpha_n\beta_n}(Y)$ is described by a symmetric tensor product of n pairs of symmetrized $sp(2)$ indices. Evaluating the tensor product yields the following decomposition

$$\Big(\underbrace{\ydiagram{2} \otimes \cdots \otimes \ydiagram{2}}_{n\ \ times}\Big)_{\rm sym} = \bigoplus_{2l+k=n} \overbrace{\underbrace{\ydiagram{8,4}}_{2l}}^{2l+2k}\,, \tag{3.25}$$

which, in fact, gives a collection of $sp(2)$ tensors of the same symmetry types as the decomposition (3.11) for $o(d-1,2)$ tensors. The only difference is that no projection on two-row tensors is needed because in two dimensions any tensor with more than two rows is identically equal to zero. Moreover, any tensor with two antisymmetric indices $X^{\alpha\beta}$ can be dualized to a scalar by virtue of the Levi-Civita tensor $\epsilon_{\alpha\beta}$ as $X^{\alpha\beta} = \dfrac{1}{2}\epsilon^{\alpha\beta}\,X$, where $X = \epsilon_{\gamma\rho}X^{\gamma\rho}$. Upon application of these rules, the decomposition (3.24) can be cast into the following form

$$F(Y) = \sum_{l=0}^{[m/2]} \sum_{k=0}^{[m/2]-l} t_{\alpha_1 \ldots\, \alpha_{2k}} \, Z_+^l \, F_{2l+k}^{\alpha_1 \ldots\, \alpha_{2k}}(Y)\,, \quad \bar{s}^{\gamma\rho} F_{2l+k}^{\alpha_1 \ldots\, \alpha_{2k}}(Y) = 0\,, \tag{3.26}$$

where the notation are introduced

$$t_{\alpha_1 \ldots \alpha_{2k}} = t_{(\alpha_1\alpha_2} \cdots t_{\alpha_{2k-1}\alpha_{2k})} \quad \text{and} \quad Z_+ = t_{\alpha\beta}t^{\alpha\beta}\,. \tag{3.27}$$

The quantities $t_{\alpha(2k)}$ and $F_{2l+k}^{\alpha(2k)}(Y)$ are rank-$2k$ symmetric $sp(2)$ tensors, and Z_+ is $sp(2)$ invariant. The expansion coefficients of $F_{2l+k}^{\alpha(2k)}(Y)$ are traceless tensors $F^{A(m-2l),\, B(m-2l-2k)}$ with the difference between lengths of first and second rows equal to $2k$. The expressions (3.26) and (3.27) provide $sp(2)$ covariant reformulation of the trace decomposition (3.12).

The above decomposition can be represented in a more compact form that will be used in the sequel. To this end let us slightly change the notation and introduce

$$F_{p,\,t}(Y) = t_{\alpha_1 \ldots\, \alpha_{2t}} F_{p,\,t}^{\alpha_1 \ldots\, \alpha_{2t}}(Y)\,, \qquad \bar{s}^{\gamma\rho} F_{p,\,t}^{\alpha_1 \ldots\, \alpha_{2t}}(Y) = 0\,, \tag{3.28}$$

where $p \geq 2t$ and $F_{p,\,t}^{\alpha(2t)}(Y)$ describe traceless $o(d-1,2)$ tensors $F^{A(p),\, B(p-2t)}$. Functions $F_{p,\,t}(Y)$ are $sp(2)$ invariant

$$T_{\alpha\beta}\, F_{p,\,t}(Y) = 0\,, \tag{3.29}$$

and subject to a generalized traceless condition

$$(\bar{s}^{\alpha\beta})^{t+1} F_{p,\,t}(Y) = 0\ . \tag{3.30}$$

The above consideration allows one to say that modulo trace contributions the quantity $F_{p,\,t}(Y)$ provides an $sp(2)$ invariant description of non-symmetric two-row $o(d-1,2)$ traceless tensors.[9] For $t=0$ one reproduces the case of traceless rectangular $o(d-1,2)$ tensors (3.16).

With the help of functions (3.28) the trace decomposition becomes now the manifestly $sp(2)$ invariant

$$F(Y) = \sum_{k=0}^{[\frac{m}{2}]} \sum_{l=0}^{[\frac{m}{2}]-k} Z_+^l\, F_{m-2l,\,k}(Y)\ . \tag{3.31}$$

In this form it admits a direct generalization to the case when an $sp(2)$ singlet $F(Y)$ is an infinite power series in the auxiliary variables, i.e.,

$$F(Y) = \sum_{m=1}^{\infty} F^{(m)}(Y)\ , \tag{3.32}$$

where $F^{(m)}$ is a polynomial of $2m-2$ order in variables Y^A_α, $F^{(m)}(tY) = t^{2m-2}F^{(m)}(Y)$. The functions $F^{(m)}(Y)$ are $sp(2)$ singlets and their expansion coefficients are two-row rectangular traceful $o(d-1,2)$ tensors. By making appropriate field redefinitions and resummations, a trace decomposition for (3.32) that generalizes (3.31) can be cast into the form

$$F(Y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \rho(k,m,n)\, Z_+^n\, F_{k,m;\,n}(Y)\ , \tag{3.33}$$

where $\rho(k,m,n)$ are some non-zero normalization coefficients and the label n in $F_{k,m;\,n}(Y)$ is introduced to mark an n-th copy of $F_{k,m}(Y)$ (3.28). It follows that traceless tensors of various symmetry types $F^{A(k),\,B(k-2m)}$ originated from $F(Y)$ (3.32) are arranged into an infinite sequence enumerated by a degree of the quantity Z_+.

4. Bilinear form

As a preamble to the following we recall that within the frame-like formulation the action functional for a HS bosonic AdS_d field (massless and partially massless) has the following schematic form

$$\mathcal{S}_2[\Omega] = \int_{\mathcal{M}^d} H^{\cdot\,\cdot\,\cdot}(V)\, \epsilon^{\cdot\,\cdot\,\cdot}{}_{M_1\ldots M_{d-4}N} E_0^{M_1}\wedge\cdots\wedge E_0^{M_{d-4}} V^N \wedge\, R^{\cdot\,\cdot\,\cdot}\wedge R^{\cdot\,\cdot\,\cdot}\ , \tag{4.1}$$

where $H^{\cdot\,\cdot\,\cdot}(V)$ are some $o(d-1,2)$ covariant coefficients which parameterize various types of index contractions between curvatures, compensators and the $(d+1)$-dimensional Levi-Civita symbol. Any such action is manifestly $o(d-1,2)$ covariant and gauge invariant

[9] More precisely, the function $F_{p,\,t}(Y)$ provides an $sp(2)$ invariant description of non-symmetric $o(d-1,2)$ tensors with a minimal number of traces involved.

with respect to the gauge transformations (2.3), (2.9). For this action to describe the correct HS field dynamics, a function $H^{\cdot\,\cdot\,\cdot}(V)$ should be fixed by the extra field decoupling condition [10 – 12, 18].

The frame-like action (4.1) fixed by the extra field decoupling condition can be reduced to the metric-like form by virtue of the partial gauge fixing [11, 18]. For the massless fields it just reproduces the Fronsdal action [1], and for the partially massless fields it yields the Lagrangian formulation of Deser and Waldron [15], and Zinoviev [17].

The formulation presented below aims to develop a Lagrangian framework that operates with infinite series of HS symmetric fields in AdS_d thus providing a starting point for the study of HS interactions. It originates from a method applied for constructing cubic interactions in AdS_5 [12, 25] and involves a description of HS fields with the help of two sets of auxiliary variables,[10] say, X and Y. The action functional is built then in the following schematic form

$$\mathcal{S}_2 = \int_{\mathcal{M}^d} \tilde{H}\Big(\frac{\partial}{\partial Y}, \frac{\partial}{\partial X}\Big)\; R(Y)\; R(X)\Big|_{X=Y=0}\,, \tag{4.2}$$

where $\tilde{H}$ is some differential operator acting on a tensor product of two HS fields. This scheme has been taken as a pattern for a description of free mixed-symmetry HS fields in AdS_d [27]. In the sequel we exploit the main idea of introducing more than one set of auxiliary variables in the form most close to that of ref. [27].

It is worth remarking that the frame-like action (4.2) built with the help of the operator acting on a tensor product of two fields resembles the typical way of introducing interactions in String Field Theory[11] (for review, see e.g. [35]). In particular, the SFT inspired approach was used in the higher spin theory for the study of cubic interactions of massless fields [36 – 39].

4.1 Triple system of auxiliary variables

Let us supplement undotted variables introduced in section 3.1 by dotted ones and define a set $Y^A_i = (Y^A_\alpha, Y^B_{\dot\alpha})$, with $\alpha, \dot\alpha = 1,2$, and $A, B = 0, \dots, d$. Also, we introduce an additional auxiliary anticommuting variable θ^A that transforms as $o(d-1,2)$ vector [27].

The following differential operators

$$\bar{s}^{ij} = \eta^{AB}\frac{\partial^2}{\partial Y^A_i \partial Y^B_j}\,, \qquad \bar{v}^i = V^A \frac{\partial}{\partial Y^A_i} \tag{4.3}$$

commute to each other and

$$\bar{\eta}^i = \eta^{AB}\frac{\partial^2}{\partial Y^A_i\,\partial\theta^B}\,, \qquad \chi = V^A\frac{\partial}{\partial\theta^A}\,, \qquad E_0 = E^A_0\frac{\partial}{\partial\theta^A} \tag{4.4}$$

anticommute to each other and commute with the set of operators (4.3). The combination

$$\Gamma = \frac{1}{(d+1)!}\,\epsilon_{A_1\,\dots\,A_{d+1}}\,\theta^{A_1}\cdots\theta^{A_{d+1}} \tag{4.5}$$

[10] The auxiliary variables can be either vectors or spinors, depending on a particular realization of HS fields. In our approach we use $o(d-1,2)$ vector variables.

[11] I am grateful to J. Buchbinder and A. Sagnotti for pointing out this to me.

is built with the help of the $(d+1)$-dimensional Levi-Civita symbol. It provides a convenient way to work with the Levi-Civita symbol being a part the action (4.1).

4.2 Bilinear symmetric form

The operators introduced above are the constituents of the following bilinear form

$$\mathcal{A}(F,G) = \int_{\mathcal{M}^d} \mathcal{H}(\bar{s},\bar{\eta},\bar{v})\,(\wedge E_0)^{d-4}\,\chi\,\Gamma \wedge F(x|\,Y) \wedge G(x|\,\dot{Y})\Big|_{Y=\dot{Y}=\theta=0}\,, \tag{4.6}$$

where $(\wedge E_0)^k$ stands for k-th exterior power of the background frame field, 2-form fields $F(x|\,Y)$ and $G(x|\,\dot{Y})$ are expansions (3.32) in undotted and dotted variables Y_α^A and $Y_{\dot\alpha}{}^A$, respectively. They are subject to the $sp(2)$ invariance conditions

$$T_{\alpha\beta}\,F(x|\,Y) = 0\,, \qquad T_{\dot\alpha\dot\beta}\,G(x|\,\dot{Y}) = 0\,. \tag{4.7}$$

The operators (4.3), (4.4), which are arguments of the function

$$\mathcal{H}(\bar{s},\bar{\eta},\bar{v}) \equiv \mathcal{H}(\bar{s}^{\alpha\beta},\bar{s}^{\dot\alpha\dot\beta},\bar{s}^{\alpha\dot\alpha},\bar{v}^{\alpha},\bar{v}^{\dot\alpha},\bar{\eta}^{\alpha},\bar{\eta}^{\dot\alpha})\,, \tag{4.8}$$

perform contractions of $o(d-1,2)$ indices inside the bilinear form: $\bar{s}^{\alpha\beta}$ and $\bar{s}^{\dot\alpha\dot\beta}$ take traces of $F(Y)$ and $G(\dot{Y})$, respectively; $\bar{s}^{\alpha\dot\alpha}$ contracts indices from $F(Y)$ and $G(\dot{Y})$; $\bar{v}^{\alpha}$ and $\bar{v}^{\dot\alpha}$ put the compensator V^A on $F(Y)$ and $G(\dot{Y})$; finally, $\bar{\eta}^{\alpha}$ and $\bar{\eta}^{\dot\alpha}$ contract indices of $F(Y)$ and $G(\dot{Y})$ with the $o(d-1,2)$ Levi-Civita symbol. Note that the operators (4.3) and (4.4) are tensors with respect to the $sp(2)$ transformations.

We require the bilinear form (4.6) to be symmetric

$$\mathcal{A}(F,G) = \mathcal{A}(G,F)\,, \tag{4.9}$$

since for $F=G$ antisymmetric terms disappear anyway. The symmetry property is equivalent to the invariance under the exchange of dotted and undotted variables inside the expression (4.6). It imposes the following constraints on the form of the function $\mathcal{H}$

$$\bar{s}^{\alpha\beta}\frac{\partial\mathcal{H}}{\partial\bar{s}^{\alpha\beta}} = \bar{s}^{\dot\alpha\dot\beta}\frac{\partial\mathcal{H}}{\partial\bar{s}^{\dot\alpha\dot\beta}}\,, \qquad \bar{v}^{\alpha}\frac{\partial\mathcal{H}}{\partial\bar{v}^{\alpha}} = \bar{v}^{\dot\alpha}\frac{\partial\mathcal{H}}{\partial\bar{v}^{\dot\alpha}}\,, \qquad \bar{\eta}^{\alpha}\frac{\partial\mathcal{H}}{\partial\bar{\eta}^{\alpha}} = \bar{\eta}^{\dot\alpha}\frac{\partial\mathcal{H}}{\partial\bar{\eta}^{\dot\alpha}}\,, \tag{4.10}$$

which mean that dotted and undotted operators enter the function $\mathcal{H}$ in equal portions. Note that the operators $\bar{s}^{\alpha\dot\beta}$ satisfy this condition automatically.

Apart from the relations (4.10) the function $\mathcal{H}$ should satisfy

$$(\bar{\eta}_\alpha\frac{\partial}{\partial\bar{\eta}_\alpha} + \bar{\eta}_{\dot\alpha}\frac{\partial}{\partial\bar{\eta}_{\dot\alpha}})\mathcal{H} = 4\,\mathcal{H}\,. \tag{4.11}$$

The reason is that the bilinear form permits only the maximal number of derivatives in the anticommuting variables θ^A, i.e., $d+1$, otherwise it is zero. This fact becomes obvious by virtue of the following relations

$$\frac{\partial}{\partial\theta^{A_1}}\cdots\frac{\partial}{\partial\theta^{A_m}}\Gamma\Big|_{\theta=0} = \begin{cases}\epsilon_{A_1\cdots A_{d+1}}\,, & m = d+1\,,\\ 0\,, & m\neq d+1\,.\end{cases} \tag{4.12}$$

Constituents of the bilinear form that contain derivatives in θ^A are those listed in (4.4). By definition of the bilinear form, there are $d-4$ derivatives coming from the frame fields E_0 and one coming from the quantity χ. It follows that the remaining four derivatives should come from the variables $\bar{\eta}_i$ what justifies the relation (4.11).

4.3 Auxiliary $sp(2)$ invariant variables

Let us now make the following observation. Consider a polynomial $F(X)$

$$F(X) = F^{\alpha_1 \dots \alpha_n}_{A_1 \dots A_n} X^{A_1}_{\alpha_1} \cdots X^{A_n}_{\alpha_n} \ , \tag{4.13}$$

where X^A_α stands for either an auxiliary variable Y^A_α or a derivative $\dfrac{\partial}{\partial Y^\alpha_A}$. The coefficients are obviously symmetric with respect to the exchange of pairs (A, α)

$$F^{\alpha_1 \dots \alpha_i \dots \alpha_j \dots \alpha_n}_{A_1 \dots A_i \dots A_j \dots A_n} = F^{\alpha_1 \dots \alpha_j \dots \alpha_i \dots \alpha_n}_{A_1 \dots A_j \dots A_i \dots A_n} \ . \tag{4.14}$$

It follows that if $sp(2)$ indices have a definite Young symmetry type then $o(d-1,2)$ indices have the same symmetry type and vice versa. As an example of using this duality just mention rectangular $o(d-1,2)$ tableaux that correspond to rectangular $sp(2)$ tableaux, or, by using the Levi-Civita symbol, to $sp(2)$ singlets, justifying in that way the $sp(2)$ invariance condition (3.9). Note also that one of dual symmetries may not be seen manifestly as it happens for $sp(2)$ singlets $F(Y)$ with the expansion coefficients written in non-manifestly covariant $sp(2)$ fashion (3.1).

The coincidence of dual symmetries makes the function $\mathcal{H}$ to be $sp(2)$ invariant. Indeed, the function $\mathcal{H}$ depends on the operators (4.3), (4.4) that perform contractions of two $o(d-1,2)$ rectangular tensors. It means precisely that $o(d-1,2)$ indices of these operators should form a group described by a rectangular tableau. Equivalently, $sp(2)$ indices of the operators should form a rectangular tableau. In other words, the function $\mathcal{H}$ should satisfy the $sp(2)$ invariance conditions

$$T_{\alpha\beta}\mathcal{H} = T_{\dot\alpha\dot\beta}\mathcal{H} = 0 \ . \tag{4.15}$$

In particular, $sp(2)$ invariance of the fields implies that $\mathcal{H}$ is defined modulo contributions proportional to $sp(2)$ generators

$$\mathcal{H} \sim \mathcal{H} + H^{\alpha\beta}T_{\alpha\beta} + H^{\dot\alpha\dot\beta}T_{\dot\alpha\dot\beta} \ , \tag{4.16}$$

where the coefficients in front of the generators are arbitrary symmetric tensors depending on the same arguments as the function $\mathcal{H}$.

The expansion coefficients of the function $\mathcal{H}$ are generically written as

$$\mathcal{H}_{\alpha_1, \dots, \alpha_{2n}; \dot\alpha_1, \dots, \dot\alpha_{2n}} \sim (\epsilon_{\alpha_1\alpha_2} \cdots \epsilon_{\alpha_{2n-1}\alpha_{2n}})\, (\epsilon_{\dot\alpha_1\dot\alpha_2} \cdots \epsilon_{\dot\alpha_{2n-1}\dot\alpha_{2n}}) \ , \tag{4.17}$$

modulo pre-factors depending on the parameter n. Both dotted and undotted indices enter in equal portions as a corollary of symmetry conditions (4.9), (4.10) and form a rectangular invariant representation of $sp(2)$ algebra. The invariance of the coefficients implies that they are tensor products of the Levi-Civita tensors $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot\alpha\dot\beta}$

$$\Big(\underbrace{\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} \otimes \cdots \otimes \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}}_{n\ times} \Big)_{\rm sym} = \overbrace{\underbrace{\begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & & & & & & & \\ \hline \ & & & & & & & \\ \hline\end{array}}_{n}}^{n} \ . \tag{4.18}$$

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Note that because of the antisymmetrization condition $\epsilon^{[\alpha\beta}\epsilon^{\gamma]\rho} = 0$, a product of $2d$ Levi-Civita tenors yields directly a Young tableau without additional symmetrizations. As a result, the form of the expansion coefficients (4.17) becomes obvious.

It is now clear how variables are arranged inside the function $\mathcal{H}$. They just form various pairings with the $2d$ Levi-Civita tenors in an $sp(2)$ invariant fashion. The idea is to single out those pairings that are elementary in the sense that all other possible pairings are their combinations. It is easy see that they are given by[12]

$$\mathsf{c}_1 = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,\bar s^{\alpha\dot\alpha}\,\bar v^{\beta}\bar v^{\dot\beta}\;, \qquad \mathsf{c}_2 = \frac{1}{4}\,\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,\bar s^{\alpha\dot\alpha}\,\bar s^{\beta\dot\beta}\;, \tag{4.19}$$

and by four more involving the trace annihilation operators $\bar s^{\alpha\beta}$ and $\bar s^{\dot\alpha\dot\beta}$,

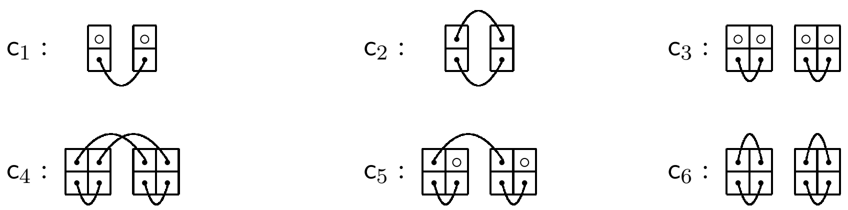
$$\begin{aligned}
\mathsf{c}_3 &= (\epsilon_{\alpha\beta}\epsilon_{\gamma\rho}\,\bar s^{\alpha\gamma}\bar v^{\beta}\bar v^{\rho})(\epsilon_{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\rho}\,\bar s^{\dot\alpha\dot\gamma}\bar v^{\dot\beta}\bar v^{\dot\rho})\;,\\
\mathsf{c}_4 &= \epsilon_{\alpha\beta}\epsilon_{\gamma\rho}\,\epsilon_{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\rho}\,\bar s^{\alpha\dot\alpha}\bar s^{\gamma\dot\gamma}\bar s^{\beta\rho}\bar s^{\dot\beta\dot\rho}\;,\\
\mathsf{c}_5 &= \epsilon_{\alpha\beta}\epsilon_{\gamma\rho}\,\epsilon_{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\rho}\,\bar s^{\alpha\dot\alpha}\bar s^{\beta\rho}\bar s^{\dot\beta\dot\rho}\bar v^{\gamma}\bar v^{\dot\gamma}\;,\\
\mathsf{c}_6 &= (\epsilon_{\alpha\beta}\epsilon_{\gamma\rho}\,\bar s^{\alpha\gamma}\bar s^{\beta\rho})(\epsilon_{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\rho}\,\bar s^{\dot\alpha\dot\gamma}\bar s^{\dot\beta\dot\rho})\;.
\end{aligned} \tag{4.20}$$

There are also some $sp(2)$ singlet pairings that involve anticommuting variables $\bar\eta_\alpha$ and $\bar\eta_{\dot\alpha}$. However, the function $\mathcal{H}$ is required to contain exactly four anticommuting variables (4.11), and the only possible combination reads

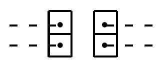
$$\bar\eta = (\epsilon_{\alpha\beta}\bar\eta^{\alpha}\bar\eta^{\beta})(\epsilon_{\dot\alpha\dot\beta}\bar\eta^{\dot\alpha}\bar\eta^{\dot\beta})\;. \tag{4.21}$$

It is obvious that $\bar\eta\bar\eta = 0$.

It is convenient to visualize the above contractions by virtue of the following pictorial representations



and

$\bar\eta$: 

Here two-cell columns are the Levi-Civita tensors, the arcs denote cross-contractions $\bar s^{\alpha\dot\alpha}$ and traces $\bar s^{\alpha\beta}, \bar s^{\dot\alpha\dot\beta}$, the circles denote $\bar v^{\alpha}$ and $\bar v^{\dot\alpha}$, and the dashed lines denote anticommuting $\bar\eta^{\alpha}$ and $\bar\eta^{\dot\alpha}$.

The coinciding dual Young symmetry types make the above graphs applicable directly for $o(d-1,2)$ tableaux. It is remarkable that two Young symmetry bases are involved simultaneously, a symmetric one, for $o(d-1,2)$ tableaux, and an antisymmetric one, for

[12]The factor $1/4$ in c_2 is introduced just for the convenience in further calculations.

$sp(2)$ tableaux. It follows that some symmetry properties not seen directly in one basis become clear in another one.

The general form of the function $\mathcal{H}$ written as a power series in the new $sp(2)$ invariant variables is

$$\mathcal{H} = H(\mathsf{c})\,\bar{\eta}\;, \qquad H(\mathsf{c}) = \sum_{k_i \geq 0}^{\infty} \xi(k_i\,; d) \prod_{i=1}^{6} (\mathsf{c}_i)^{k_i}\;, \tag{4.22}$$

where $\xi(k_i\,; d)$ are some d-dependent coefficients.

5. Action for symmetric HS fields

So far we have elaborated the framework that utilizes $sp(2)$ doublets of auxiliary vector variables for a description of non-symmetric higher-rank tensors. Now we are in a position to adjust it for Lagrangian formulation of HS field dynamics along the lines discussed in the beginning of the previous section.

Prior to continue let us make a comment that the $sp(2)$ invariance condition does not specify particular lengths of Young tableau and just requires it to be a rectangular block. In particular, it allows one to consider infinite sets of the same symmetry type tensors on equal footing. On the contrary, in the manifest antisymmetric basis for rectangular Young tableaux Howe dual algebra becomes $sl(m)$, where m is a length of the uppermost row [30, 21]. However, the use of antisymmetric basis is inconvenient because one should introduce then an infinite chain of Howe dual algebras that correspond to an infinite set of Young tableaux with increasing lengths.

5.1 Action functional: general properties

Let us introduce the $sp(2)$ invariant 1-form gauge field

$$\Omega(x|\,Y) = \mathrm{d}x^{\underline{n}}\,\Omega_{\underline{n}}(x|\,Y)\;, \qquad T_{\alpha\beta}\Omega(x|\,Y) = 0\;. \tag{5.1}$$

Frame-like higher spin gauge fields are then identified with the expansion coefficients of $\Omega(x|\,Y)$ with respect to the auxiliary variables. An irreducible field (massless or partially massless) of a given spin $s' = s$ and depth $t' = s - 2t$ appears in $\Omega(x|\,Y)$ in infinitely many copies because the trace decomposition (3.33) requires

$$\Omega(Y|\,x) = \sum_{n,\,s,\,t=0}^{\infty} \rho(s,t,n)\, Z_+^n\, \Omega_{s,\,t;\,n}(Y|\,x)\;, \tag{5.2}$$

where $\rho(s,t,n)$ are some normalization coefficients. Note that the parameter t is necessarily even, and therefore the partially massless fields appearing in (5.2) are not arbitrary. On the contrary, massless fields ($t = 0$) appear with any value of spin s from zero to infinity.

The field (5.1) is a gauge connection of the HS algebra $hc(1|2:[d-1,2])$, frequently regarded as the off-shell algebra [20−22, 31]. The presence of partially massless states in the spectrum violates the unitarity already on the free field level. To get rid of them a procedure is required that allows one to do this consistently in the sense that the remaining fields should be organized in a multiplet of some HS algebra. In d dimensions it precisely corresponds to the factoring out an ideal of $hc(1|2:[d-1,2])$ generated by traces of the field (5.1). It gives rise to the on-shell algebra $hu(1|2:[d-1,2])$ of [20]. The resulting theory will be described by (5.2) with all $\Omega_{s,\,t;\,n}(Y|\,x)$ at $t\neq 0, n\neq 0$ set to zero. In $d=5$ a weaker truncation is possible which drops out all partially massless fields but retains massless fields in infinitely many copies. The final set of fields corresponds to the spectrum of $5d$ algebra $hu(1,1|8)$ [32 – 34, 12].

Now the HS action functional can be defined by virtue of the bilinear form (4.6) in the following way

$$\mathcal{S}_2[\Omega] = \frac{1}{2}\,\mathcal{A}(R,R)\;, \tag{5.3}$$

where the linearized curvatures (2.2), (2.8)

$$R(Y|\,x) = D_0\Omega(Y|\,x) \qquad \text{and} \qquad R(\dot{Y}|\,x) = D_0\Omega(\dot{Y}|\,x) \tag{5.4}$$

are associated with gauge fields (5.2) and invariant under the transformations (2.3), (2.9)

$$\delta\Omega(Y|\,x) = D_0\varepsilon(Y|\,x) \qquad \text{and} \qquad \delta\Omega(\dot{Y}|\,x) = D_0\varepsilon(\dot{Y}|\,x)\;. \tag{5.5}$$

Both curvatures and gauge parameters naturally inherit the property of $sp(2)$ invariance

$$\begin{aligned} T_{\alpha\beta}\,R(Y|\,x) = T_{\dot{\alpha}\dot{\beta}}\,R(\dot{Y}|\,x) = 0\;,\\ T_{\alpha\beta}\,\varepsilon(Y|\,x) = T_{\dot{\alpha}\dot{\beta}}\,\varepsilon(\dot{Y}|\,x) = 0\;. \end{aligned} \tag{5.6}$$

As discussed above, the gauge field $\Omega(Y|\,x)$ describes an infinite sum of irreducible fields in infinitely many copies. However, it should not be taken for granted that the action functional (5.3) describes a direct sum of the actions for irreducible fields. In other words, an additional condition should be imposed that makes the action (5.3) diagonal. Such a condition requires all the cross-terms containing products of fields $\Omega_{s,\,t;\,m}(x)$ and $\Omega_{s,\,t;\,n}(x)$ for $m\neq n$ to vanish

$$\mathcal{S}_2[\Omega] = \sum_n\sum_{s,\,t}\,\chi(s,\,t;\,n)\,\mathcal{S}_2[\Omega_{s,\,t;\,n}]\;, \tag{5.7}$$

where $\chi(s,\,t;\,n)$ are some normalization coefficients. The diagonalization condition will be considered elsewhere [26].

Another condition to be imposed on the action $\mathcal{S}_2[\Omega]$ is the decoupling of extra fields [10 – 12, 18]. It requires that the extra fields should enter the action through total derivatives, i.e.,

$$\frac{\delta\mathcal{S}_2[\Omega]}{\delta\Omega^{extra}} \equiv 0\;, \tag{5.8}$$

and means that the action depends non-trivially on the physical and the auxiliary fields only.

The action functional (5.3) subject to the diagonalization condition (5.7) and the extra field decoupling condition (5.8) describes both massless and partially massless free symmetric fields and each field appears in infinitely many copies. These two conditions fix the action unambiguously up to an arbitrary normalization coefficient in front of a given spin action $\mathcal{S}_2[\Omega_{s,\,t;\,n}]$ (5.7).

5.2 Action functional: a non-degenerate set of fields

In this section we study the general form of the action functional which describes a non-degenerate set of symmetric fields, that is each field enters in a single copy. In this case the diagonalization condition (5.7) is relaxed.

Let us suppose that all irreducible fields enter $\Omega(Y|\,x)$ in a single copy and for this occasion introduce a notation $\Omega_{s,\,t}(Y|\,x) \equiv \Omega_{s,\,t;\,0}(Y|\,x)$. As discussed in section 3.2, the field $\Omega_{s,\,t}(Y|\,x)$ satisfies the trace condition

$$(\bar{s}^{\alpha\beta})^{t+1}\,\Omega_{s,\,t}(Y|\,x) = 0\;. \tag{5.9}$$

Because traces enter $\Omega_{s,\,t}(Y|\,x)$ in a totally symmetric combination (3.28), it follows that a non-symmetric combination of the annihilation trace operators acts on it by zero

$$(\epsilon_{\alpha\beta}\epsilon_{\gamma\rho}\,\bar{s}^{\alpha\gamma}\bar{s}^{\beta\rho})\,\Omega_{s,\,t}(Y|\,x) = 0\;. \tag{5.10}$$

Indeed, the symmetry types of components arising in a tensor product of the trace annihilation operators $\bar{s}^{\alpha\beta}$ are precisely described by the decomposition (3.25) that up to pre-factors takes now the following form

$$(\bar{s}^{\alpha\beta})^n \sim \sum_{2l+k=n} \bar{s}^{\alpha_1\ldots\alpha_{2k}}\,Z_-^l\;, \tag{5.11}$$

where the notation are introduced

$$\bar{s}^{\alpha_1\ldots\alpha_{2k}} = \bar{s}^{(\alpha_1\alpha_2}\ldots\bar{s}^{\alpha_{2k-1}\alpha_{2k})} \qquad \text{and} \qquad Z_- = \epsilon_{\alpha\beta}\epsilon_{\gamma\rho}\,\bar{s}^{\alpha\gamma}\bar{s}^{\beta\rho}\;. \tag{5.12}$$

Then the result is that the function $H(\mathsf{c})$ (4.22) becomes independent of the variable c_6, i.e.,

$$\frac{\partial H}{\partial \mathsf{c}_6} = 0\;, \tag{5.13}$$

since $\mathsf{c}_6 = Z_-\dot{Z}_-$ acts trivially on the fields $\Omega_{s,\,t}(Y|\,x)$ and $\Omega_{s,\,t}(\dot{Y}|\,x)$. Another consequence of the relation (5.10) is that modulo terms containing non-symmetric combinations of traces, the following algebraic constraint takes place

$$\mathsf{c}_5^2 = \mathsf{c}_3\mathsf{c}_4\;. \tag{5.14}$$

As a result, the function H becomes linear in variable c_5, that is

$$H = H_1(\mathsf{c}) + \mathsf{c}_5\,H_2(\mathsf{c})\;, \qquad \frac{\partial H_1(\mathsf{c})}{\partial \mathsf{c}_5} = \frac{\partial H_2(\mathsf{c})}{\partial \mathsf{c}_5} = 0\;. \tag{5.15}$$

We see that the function $H(\mathsf{c})$ corresponding to the infinite series of irreducible symmetric fields (each in a single copy) depends essentially on four variables. This fact is in agreement with the component form of HS actions originally elaborated in [11, 12, 18]. Indeed, before the extra field decoupling condition is imposed, the coefficients in the action are parameterized by four numbers, $a(s,t;m,l)$, where s and t define a size of traceless $o(d-1,2)$ Young tableau, m and l correspond to the numbers of the compensator vectors contracted with first and second rows, respectively. The massless fields are described by rectangular tableaux and therefore one is left with just two parameters, s and m. Within our approach it corresponds to

$$H = H(\mathsf{c}_1, \mathsf{c}_2)\,, \tag{5.16}$$

where all traceful contributions are eliminated because the variables involving trace operators act trivially, $\partial H/\partial \mathsf{c}_i = 0$, $i = 3, 4, 5, 6$.

5.3 Action for massless symmetric fields

We have emphasized earlier that in order to have unitary dynamics partially massless fields are required to decouple from the whole system of fields. In this section we drop out all partially massless fields by hand and consider massless fields only. Such a truncation is consistent on the free field level. The resulting set of fields with spins $0 \leq s < \infty$ form the multiplet of symmetric massless fields of the algebra $hu(1|2\!:\![d-1, 2])$ [21]. Note that lower spin fields with $s \leq 1$ do not admit a frame-like Lagrangian form (4.1) and should be described by standard Klein-Gordon and Maxwell actions.

The action for a single massless field of spin s is written down as

$$\mathcal{S}_2[\Omega_{s,\,0}] = \frac{1}{2}\int_{\mathcal{M}^d} \mathcal{H}\,(\wedge E_0)^{d-4}\,\chi\,\Gamma \wedge R_{s,\,0}(x|\,Y) \wedge R_{s,\,0}(x|\,\dot{Y})\Big|_{Y=\dot{Y}=0}\,, \tag{5.17}$$

where

$$\mathcal{H} = H(\mathsf{c}_1, \mathsf{c}_2)\,\bar{\eta}\,, \qquad H(\mathsf{c}_1, \mathsf{c}_2) = \frac{1}{(s-1)}\sum_{m=0}^{s-2} \xi(m; d, s)\;\mathsf{c}_1^{s-m-2}\,\mathsf{c}_2^{m}\,. \tag{5.18}$$

Here $\xi(m; d, s)$ are arbitrary real coefficients parameterized by three numbers, fixed $s \geq 2$ and $d \geq 4$, and running m.

To impose the extra field decoupling condition we make use of the method elaborated in ref. [27]. Namely, one observes that in order to have a manifest gauge invariance, the action is always written down with the extra fields, at least formally.[13] On the contrary, the manifestly gauge invariant field equations satisfying the decoupling condition are easier to find since they depend on two fields only, physical and auxiliary ones. The idea is then to reconstruct the action from the known field equations by requiring them to follow from the action. To perform a reconstruction the cohomological technique based on the so-called $\mathcal{Q}$-complex was elaborated [27]. Here we find the action from the field equations by another method, which turns out to be more appropriate for our purposes.

The field equations that follow from a variation of the action for massless fields and satisfy the extra field decoupling condition have the form

$$\delta\mathcal{S}_2[\Omega_{0,\,s}] = \int_{\mathcal{M}^d} \mathcal{E}(\bar{s}, \bar{\eta}, \bar{v})\,(\wedge E_0)^{d-3}\,\chi\,\Gamma \wedge R_{s,\,0}(x|\,Y) \wedge \delta\Omega_{s,\,0}(x|\,\dot{Y})\Big|_{Y=\dot{Y}=0} = 0 \tag{5.19}$$

and the function $\mathcal{E}(\bar{s}, \bar{\eta}, \bar{v})$ is defined in the $sp(2)$ invariant way as

$$\mathcal{E} = (\bar{\pi}\dot{\bar{\tau}} - \dot{\bar{\pi}}\bar{\tau})\,T(\mathsf{c}_1)\,, \tag{5.20}$$

[13] Having decoupled extra fields, the action can be cast into a minimal form with just two fields, the physical and the auxiliary, but the residual gauge invariance is implicit [9, 40].

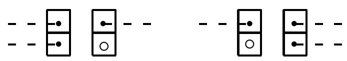
where variables $\bar{\pi}$ and $\bar{\tau}$ and their dotted cousins are the following $sp(2)$ singlet pairings

$$\bar{\pi} = \epsilon_{\alpha\beta}\bar{\eta}^{\alpha}\bar{\eta}^{\beta}\ , \quad \bar{\tau} = \epsilon_{\alpha\beta}\bar{v}^{\alpha}\bar{\eta}^{\beta}\ ,$$
$$\dot{\bar{\pi}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\eta}^{\dot{\alpha}}\bar{\eta}^{\dot{\beta}}\ , \quad \dot{\bar{\tau}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{v}^{\dot{\alpha}}\bar{\eta}^{\dot{\beta}}\ . \tag{5.21}$$

Note that $\bar{\tau}\bar{\tau} = 0$ and $\dot{\bar{\tau}}\dot{\bar{\tau}} = 0$, and $\bar{\pi}\dot{\bar{\pi}} = \bar{\eta}$ (cf. (4.21)). The function $T(\mathsf{c}_1)$ is an arbitrary polynomial, which in the case of a single massless field becomes a monomial

$$T(\mathsf{c}_1) = \mathsf{c}_1^{s-2}\ . \tag{5.22}$$

The extra field decoupling condition is automatically satisfied by the field equations (5.19). Indeed, according to formula (2.5) the physical and the auxiliary fields are contained in the original field $\Omega_{s,\,0}(x|\,\dot{Y})$ contracted with $(s-2)$ compensators. It explains the appearance of the function $T(\mathsf{c}_1)$ (5.22). Then, the first term of (5.20) contains maximal possible number $(s-1)$ of compensators contracted with $\delta\Omega_{s,\,0}(x|\,\dot{Y})$ and therefore, as discussed in section 2, corresponds to the variation with respect to the physical field. Analogously, the second term contains $(s-2)$ compensators contracted with $\delta\Omega_{s,\,0}(x|\,\dot{Y})$ and therefore, corresponds to the variation with respect to the auxiliary field. In both the contractions, the remaining index of $R_{s,\,0}(x|\,Y)$ or $\delta\Omega_{s,\,0}(x|\,\dot{Y})$ is contracted with the $o(d-1,2)$ Levi-Civita tensor. The terms $(\bar{\pi}\dot{\bar{\tau}} - \dot{\bar{\pi}}\bar{\tau})$ in (5.20) can be described by the following pictorial representation



where left and right columns correspond to the curvature and the field variation.

Let us now obtain the variation of the action and equate it to the field equations defined by (5.19) and (5.20). It results in the equation

$$\left(2(d-3) + 4\mathsf{c}_1\frac{\partial}{\partial\mathsf{c}_1} - 4\mathsf{c}_2\frac{\partial}{\partial\mathsf{c}_1}\right)H(\mathsf{c}_1,\mathsf{c}_2) = T(\mathsf{c}_1)\ . \tag{5.23}$$

It is solved by the following integral expression (see appendix A.1 for more details)

$$H(\mathsf{c}_1,\mathsf{c}_2) = \frac{1}{4}\int_0^1 \mathrm{d}t\ t^{(d-5)/2}\ \exp\left(\frac{1-t}{t}\ \mathsf{c}_2\frac{\partial}{\partial\mathsf{c}_1}\right)\ T(t\mathsf{c}_1)\ . \tag{5.24}$$

This formula trivially generalizes to the case of an arbitrary polynomial function $T(\mathsf{c}_1)$ thus giving rise to a direct sum of spin-s actions with $2 \leq s \leq \infty$.

After some simple algebra (see appendix A.2) one obtains the expansion coefficients (5.18) expressed in terms of the beta functions

$$\xi(m;d,s) = \frac{B(m+1, s-m-1+(d-5)/2)}{B(m+1, s-m-1)}\ . \tag{5.25}$$

This answer can be compared with the coefficients arising in the component form of the action. To this end, one violates manifest $sp(2)$ invariance and introduces instead of c_1 and c_2 new variables $x_1 = \bar{s}^{1\dot{1}}\bar{v}^2\bar{v}^{\dot{2}}$ and $x_2 = \bar{s}^{1\dot{1}}\bar{s}^{2\dot{2}}$ that perform row-to-row contractions of two rectangular tableaux. A change of variables done inside the action gives the expression

$$c_1^m c_2^n \sim 2^n(m+1)(m+n+1)x_1^m x_2^n\ , \tag{5.26}$$

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where $\sim$ means that the equality is valid up to terms proportional to Young symmetrizers (3.3) that trivialize when acting on Young tableaux. Then, by using formulas of appendix A.2, the function (5.25) can be cast into the more traditional form with (double) factorials known from ref. [12]

$$\zeta(m;d,s) = \zeta(d,s)\,\frac{(s-m-1)(d-5+2(s-m-2))!!}{(s-m-2)!}\,, \tag{5.27}$$

where $\zeta(d,s)$ is an overall factor in front of the spin-s action.

In conclusion, let us make a comment that for a single massless fields the Howe dual $sp(2)$ algebra is enhanced to $sp(4)$ but it seems that within our approach $sp(4)$ does not play any essential role. Therefore it would be interesting to see more deeper implementation of $sp(4)$ symmetry for the frame-like formulation of symmetric fields. For example, within the first-quantized BRST approach to higher spin dynamics [13] Howe dual $sp(4)$ algebra appears as an extension of first-class constraint algebra that describes classical mechanics of a particle with the spin degrees of freedom.

6. Conclusions

We offered the new perspective on using $sp(2)$ symmetry in the Lagrangian HS dynamics of bosonic symmetric fields and elaborated on the idea of introducing $sp(2)$ invariant variables. The whole formulation is designed to deal with infinite multiplets of fields naturally appearing as the HS gauge connections. The present paper can be considered as a first step towards the study of Lagrangian form of the HS interactions. Having this in mind let us now summarize our results.

- We have elaborated more on HS fields written in terms of $sp(2)$ auxiliary variables and, in particular, studied the trace decompositions that allows one to control contributions of massless and partially massless symmetric fields. Higher rank tensors described as polynomials of auxiliary commuting variables Y^A_α are in fact elements of the $*$-product algebra generated by $Y^A_\alpha * Y^B_\beta - Y^B_\beta * Y^A_\alpha = \epsilon_{\alpha\beta}\eta^{AB}$. In this form these higher rank tensors are naturally appear as the HS connections [20].

- We have introduced the bilinear form $\mathcal{A}(F,G)$ defined on arbitrary $sp(2)$ singlet fields $F(Y\,|x)$ and $G(\dot{Y}\,|x)$. It serves as the main building block of the HS action functionals to be considered both on the free field and the interaction levels. The main novel ingredient is the use of $sp(2)$ singlet variables that allows one to avoid dealing with the explicit Young symmetrizers inside the bilinear form and thus considerably simplify calculations.

- Within our approach we have analyzed the general properties of the quadratic higher spin actions and, in particular, considered the action describing a single symmetric field. We have explicitly built the action for free massless symmetric fields and reproduced the well-known expression for coefficients of the component form of HS action originally obtained in [12]. Our answer is given as an integral of the exponential

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operator and we expect that this form of coefficients is suitable for dealing with $*$-product when studying the HS interactions.

- Summarizing the above, our main result is that we have brought together formulations used previously for the HS algebra [20] and the HS action functionals [12, 27] and provided for them a unified framework.

Within our approach we could also reproduce the component form of HS action for partially massless fields [18]. Let us stress once again that partially massless fields appearing in the trace decompositions of traceful rectangular fields have an even difference of lengths therefore their spins s and depths t are not arbitrary.

To conclude, let us mention the following directions for the further study:

- A Lagrangian form of non-linear dynamics of partially massless and massless fields interacting between themselves and with the gravity. It is expected that such a theory should be governed by the "off-shell" algebra $hc(1|2\!:\![d-1,2])$ [21].
- A Lagrangian form of non-linear dynamics of massless fields only. This theory is based on the "on-shell" algebra $hu(1|2\!:\![d-1,2])$, which is the quotient of the "off-shell" algebra [20, 21]. On the level of the equations of motion it was formulated in ref. [20]. To develop a Lagrangian formulation we suggest to use the projection technique as we described in the Introduction.

Hopefully, the interaction problems listed above could be explicitly analyzed at least in the cubic approximation [26].

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A. Details of calculations

A.1 The extra field decoupling condition

The extra field decoupling condition results in the partial differential equation (5.23) which is a particular case of

$$\left(\alpha+\beta \mathsf{c}_1\frac{\partial}{\partial \mathsf{c}_1}+\gamma \mathsf{c}_2\frac{\partial}{\partial \mathsf{c}_1}\right)H(\mathsf{c}_1,\mathsf{c}_2)=T(\mathsf{c}_1,\mathsf{c}_2)\,,$$

where α, β and γ are some constants and $T(\mathsf{c}_1, \mathsf{c}_2)$ is a given polynomial function. To find the function $H(\mathsf{c}_1, \mathsf{c}_2)$ we solve this equation in two steps. First, one observes that the following identity is valid

$$\exp\left(\rho\,\mathsf{c}_2\frac{\partial}{\partial c_1}\right)N_1 - N_1\exp\left(\rho\,\mathsf{c}_2\frac{\partial}{\partial \mathsf{c}_1}\right) = \rho\mathsf{c}_2\frac{\partial}{\partial c_1}\exp\left(\rho\,\mathsf{c}_2\frac{\partial}{\partial \mathsf{c}_1}\right),$$

where ρ is an arbitrary constant and $N_1 = \mathsf{c}_1\dfrac{\partial}{\partial \mathsf{c}_1}$ is the Euler operator. By making use of this identity the original equation is cast into the form

$$\left(\frac{\alpha}{\beta} + N_1\right)\tilde{H}(\mathsf{c}_1, \mathsf{c}_2) = \frac{1}{\beta}\,\tilde{T}(\mathsf{c}_1, \mathsf{c}_2)\;,$$

where the tildes mark functions transformed as

$$\tilde{F}(\mathsf{c}_1, \mathsf{c}_2) = \exp\left(-\frac{\gamma}{\beta}\,\mathsf{c}_2\frac{\partial}{\partial \mathsf{c}_1}\right)F(\mathsf{c}_1, \mathsf{c}_2)\;.$$

By stretching variable $\mathsf{c}_1 \to t\mathsf{c}_1$, it is easy to see that the function

$$\tilde{H}(\mathsf{c}_1, \mathsf{c}_2) = \int_0^1 dt\, t^{\alpha/\beta-1}\,\tilde{T}(t\mathsf{c}_1, \mathsf{c}_2)$$

provides a solution to the last equation. By making a pullback map one solves the original equation as

$$H(\mathsf{c}_1, \mathsf{c}_2) = \frac{1}{\beta}\int_0^1 dt\, t^{\alpha/\beta-1}\exp\left(\frac{\gamma}{\beta}\frac{t-1}{t}\,\mathsf{c}_2\frac{\partial}{\partial \mathsf{c}_1}\right)T(t\mathsf{c}_1, \mathsf{c}_2)\;.$$

By substituting particular values of parameters $\alpha = 2(d-3)$, $\beta = 4$ and $\gamma = -4$, and $T = T(\mathsf{c}_1)$ one reproduces formula (5.24).

A.2 The beta and gamma functions

We use the following representations for the beta and gamma functions

$$B(m,n) = \int_0^1 dt\, t^{m-1}(1-t)^{n-1}\;, \qquad \mathrm{Re}[m] > 0,\;\; \mathrm{Re}[n] > 0\;.$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\;.$$

$$\Gamma(p/2+1) = \sqrt{\pi}\frac{p!!}{2^{(p+1)/2}} \qquad (\text{odd}\quad p)\;, \tag{A.1}$$

$$\Gamma(p/2+1) = \frac{p!!}{2^{p/2}} \qquad (\text{even}\quad p)\;. \tag{A.2}$$

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